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On the evaluation of Dirac traces

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Abstract. Formulae are given for the most compact evaluation of traces of Dirac matrices in the case that a certain matrix $\not{a} = a^{\mu}\gamma_{\mu}$ and/or the combination $(\not{p} + m)$ occur several times. The formulae should also be useful in connection with symbolic computation programs.

1. Introduction

Calculating traces of Dirac matrices is a necessary ingredient of many calculations in high-energy physics. Whereas traces consisting of only a few matrices are immediately written down and traces of intermediate length are conveniently dealt with by symbolic computation programs (e.g. REDUCE by Hearn 1973) there is a problem with longer traces. Even if the result is comparably simple, the available symbolic computation programs may have to handle a huge number of terms at an intermediate stage which eventually exceeds the storage capacity of the computer. Or, more likely, the result becomes simple only due to an appropriate form of representation. This is hard to extract from a term-by-term output of a symbolic computation program.

In this paper we give formulae for two frequently occurring cases where an essential simplification can be achieved. In §§ 2 and 3 we give formulae for the case that a certain vector a^{μ} and/or the matrix $\not{p}+m$ occur repeatedly. In § 4 we make some remarks concerning the application of these formulae and give an example.

The conventions of Bjorken and Drell (1965) are used throughout.

2. Traces containing a certain vector repeatedly

The problem of calculating traces seems to be solved by the following formula:

$$\operatorname{Tr} \not{a}_1\not{a}_2 \ldots \not{a}_{2n} = 4 \sum_{\substack{1=i_1<i_2<\ldots<i_n \\ i_k<j_k}} (-1)^P (a_{i_1}a_{j_1})(a_{i_2}a_{j_2}) \ldots (a_{i_n}a_{j_n}) \qquad (1)$$

where P is the number of permutations necessary to obtain the arrangement of a's on the right-hand side of equation (1) from that of the left-hand side. The trace displayed in equation (1) consists of $(2n-1)!!$ terms which are all different provided all a_i are different from each other. If, however, some of the vectors a_i are equal, extensive cancellations reduce the number of terms considerably. The formula given below exhibits for that case immediately the final expansion in powers of a^2.

We introduce the notation (which is consistent with the scalar product of two vectors)

$$(a_1 a_2 \ldots a_{2n}) = \tfrac{1}{4}\,\mathrm{Tr}\, \not{a}_1 \not{a}_2 \ldots \not{a}_{2n}.$$

Let

$$B^{(i)} = \not{b}_{l_{i-1}+1} \not{b}_{l_{i-1}+2} \ldots \not{b}_{l_i} \qquad (l_0 = 0,\ l_i + 1 \leq l_{i+1})$$

and let $B^{(1)} \ldots B^{(n)};\, b_{i_1} \ldots b_{i_k}$ denote the matrix obtained from $B^{(1)} \ldots B^{(n)}$ by omitting $\not{b}_{i_1} \ldots \not{b}_{i_k}$ without changing the order of the remaining matrices. We then have the following theorem:

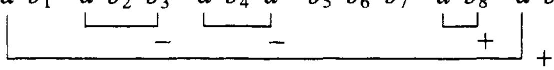
$$\begin{aligned}(aB^{(1)}aB^{(2)} \ldots aB^{(n)}) &= \sum_{\nu=0}^{[n/2]} (a^2)^\nu \max\{1, 2^{n-2\nu-1}\} \sum{}' (-1)^P (ab_{i_1})(ab_{i_2}) \ldots (ab_{i_{n-2\nu}}) \\ &\quad \times (B^{(1)} \ldots B^{(n)};\, b_{i_1} \ldots b_{i_{n-2\nu}}). \end{aligned} \tag{2}$$

The second sum (indicated by a prime) extends over all $i_1 < i_2 < \ldots < i_{n-2\nu}$ subject to the condition that for all $k < 1$, $k, l = 1, \ldots, n-2\nu$.

$$\begin{aligned}&\text{If } b_{i_k} \in B^{(r_k)} \text{ and } b_{i_l} \in B^{(r_l)}, \\ &\text{then } r_l - r_k \text{ is an odd integer:} \\ &r_l - r_k = 1 + 2\rho_{lk}, \qquad \rho_{lk} = 0, 1, 2, \ldots . \end{aligned} \tag{3}$$

The sign $(-1)^P$ of a particular term is obtained as follows. For each scalar product, a^2 or (ab_i), draw an arc connecting an a with an a or an a with b_i in such a way that the arcs do not overlap, but otherwise arbitrarily. Then each arc contributes a factor of $(-1)^N$ to the sign where N is the number of matrices enclosed by the arc. For example, with the trace

$$(a\ b_1\ \ a\ b_2\ b_3\ \ a\ b_4\ a\ \ b_5\ b_6\ b_7\ \ a\ b_8\ \ a\ b_9\ b_{10})$$



the term proportional to $(a^2)^2 (a\,b_3)(a\,b_8)(b_1 b_2 b_4 b_5 b_6 b_7 b_9 b_{10})$ contributes with a positive sign as indicated.

Equation (2) and the accompanying explanations may look more complicated than they really are. The crucial point is formulated in equation (3). It means that, for example, the trace $(ab_1\ ab_2b_3\ ab_4b_5\ ab_6)$ contains terms proportional to $a^2(ab_1)(ab_2)$, $a^2(ab_1)(ab_6)$, but not $(a^2)(ab_1)(ab_4)$, $(ab_1)(ab_2)(ab_3)(ab_6)$. The essential achievement of equation (2) is that all terms are really different from each other provided all b_i are different.

Equation (2) is proved by means of the Pfaffian formalism advocated by Caianiello (1973). Note that the right-hand side of equation (1) is just the definition of a Pfaffian. There is an expansion theorem for Pfaffians in terms of lower-order Pfaffians. Equations (2) and (3) follow readily from this theorem if the expansion is done with respect to those elements which are not equal. This turns out to yield an expansion in powers of a^2.

3. Traces containing $\not{p}+m$ repeatedly

For this case a useful device has been given by Caianiello and Fubini (1952) which we review here for our purpose. Let $\not{P}_i=\not{p}_i+m_i$; then

$$\begin{aligned}\not{P}_1\ldots\not{P}_{2n}&=\not{P}_1(-\mathrm{i}\gamma_5)(\mathrm{i}\gamma_5)\not{P}_2\not{P}_3(-\mathrm{i}\gamma_5)(\mathrm{i}\gamma_5)\not{P}_4\ldots\not{P}_{2n-1}(-\mathrm{i}\gamma_5)(\mathrm{i}\gamma_5)\not{P}_{2n}\\&=\not{Q}_1\hat{\not{Q}}_2\ldots\not{Q}_{2n-1}\hat{\not{Q}}_{2n}\end{aligned}\tag{4}$$

where

$$\begin{aligned}&\not{Q}_i=\Gamma_\alpha p^\alpha, && \hat{\not{Q}}_i=\Gamma_\alpha\hat{p}^\alpha && (\alpha=0,1,\ldots,4)\\&\Gamma_\mu=\mathrm{i}\gamma_5\gamma_\mu && (\mu=0,1,\ldots,3), && \Gamma^4=-\Gamma_4=\mathrm{i}\gamma_5\\&\{\Gamma_\alpha,\Gamma_\beta\}=2g_{\alpha\beta}, && g_{\alpha\beta}=(+----)\\&p^\alpha=(p^\mu,m), && \hat{p}^\alpha=(p^\mu,-m).\end{aligned}\tag{5}$$

Similarly

$$\not{P}_1\ldots\not{P}_{2n+1}=\not{Q}_1\ldots\hat{\not{Q}}_{2n}\not{Q}_{2n+1}\mathrm{i}\gamma_5.\tag{6}$$

In equations (4) and (6) the $\not{P}_i$, consisting of a Dirac matrix and a scalar, are replaced by the five component objects Q_i such that the matrices Γ fulfil the same commutation relations as the γ's.

The following substitution rule is a simple consequence of equation (4) $(n+l_n=\text{even})$

$$\mathrm{Tr}(\not{p}_1+m_1)B^{(1)}(\not{p}_2+m_2)B^{(2)}\ldots(\not{p}_n+m_n)B^{(n)}=\mathrm{Tr}\,\not{p}_1B^{(1)}\ldots\not{p}_nB^{(n)}\big|_{(p_ip_j)\to(p_ip_j)+(-1)^{\epsilon_{ij}}m_im_j}.\tag{7}$$

Here ϵ_{ij} is the number of γ-matrices between $\not{p}_i$ and $\not{p}_j$ on the right-hand side of equation (7). If one wants to take advantage of the fact that some p_i are equal, the simple substitution rule (7) becomes ambiguous. In that case one has to make explicit use of the five-component formalism described in equations (4)–(6). If for example all p_i are equal, one has only to discriminate between $\not{Q}$ and $\hat{\not{Q}}$; $Q^2=\hat{Q}^2=(p^2-m^2)$, but $\{\not{Q},\hat{\not{Q}}\}=2(p^2+m^2)$. Equations (4)–(6) can now be combined with equations (2)–(3), since the proof of the latter only made use of the commutation relations $\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu}$ which hold also for the Γ's. If things can be arranged such that only Q or $\hat{Q}$ occurs, the evaluation becomes particularly simple.

4. Remarks and an example

As usual, explicit appearance of the matrix γ_5 does not fit very well in this scheme†. If it is possible to construct from the occurring vectors four orthogonal ones a_μ, b_μ, c_μ, d_μ, say, it may prove most convenient to replace γ_5 according to

$$\gamma_5=-\mathrm{i}(\epsilon_{\alpha\beta\gamma\delta}a^\alpha b^\beta c^\gamma d^\delta)^{-1}\not{a}\not{b}\not{c}\not{d}$$

† This may look surprising with respect to equations (4)–(6). However, only traces of an even number of five-component Q's are easy to evaluate. Both (4) and (6) incorporate an even number. Traces of an odd number do not vanish and there are no simple formulae for them.

If there are less than four linearly independent vectors, traces involving γ_5 vanish anyway.

Unfortunately, equations (4)–(7) cannot be combined with the algorithm due to Kahane (1968) or its generalization by Chisholm and Hearn (Chisholm 1972), which evaluates such products as $\gamma_\mu B^{(1)}\gamma_\nu B^{(2)}\gamma^\mu B^{(3)}\gamma^\nu B^{(4)}$. This is because the five-component Q's have no definite parity. Equation (7) can also be proved without the five-component formalism. Then, the incompatibility arises from the fact that a term (p_ip_j) can result either from a direct contraction or from $(p_i\gamma^\mu)(p_j\gamma_\mu)=p_i^\mu p_{j\mu}$. In the first case one should substitute according to (7), but this is not so in the second. Hence, the substitution rule cannot apply. Equation (2) may be profitably used, however, after the application of Kahane's algorithm.

Although the usefulness of the formulae given above is limited by their incompatibility with Kahane's algorithm they still offer in many cases a tremendous simplification. These include calculations in external laser fields (where a lightlike vector occurs at every vertex) and the case where the vector propagator is not simply proportional to $g_{\mu\nu}$. In the latter case Kahane's algorithm does not apply at all, but a systematic simplification can be achieved by means of the formulae given above.

As an example, making use of equations (2)–(3) and (7), the trace

$$\tfrac{1}{4}\,\mathrm{Tr}(\not p+m)\not a_1(\not p+m)\not a_2\ldots(\not p+m)\not a_8$$

of a product of 16 γ-matrices is immediately written down to yield (we use the abbreviations $(pa_i)=(pi)$, $(a_1\,a_2\,a_3\,a_4)=(1\;2\;3\;4)$, etc, CP = cyclic permutations:

$$\begin{aligned}(p^2-m^2)^4(1\ldots 8)-2(p^2-m^2)^3[(p1)(p2)(3\ldots 8)+\text{CP}+(p1)(p4)(2\;3\;5\ldots 8)+\text{CP}]\\ +8(p^2-m^2)^2[(p1)(p2)(p3)(p4)(5\;6\;7\;8)+\text{CP}\\ +(p1)(p2)(p3)(p6)(4\;5\;7\;8)+\text{CP}+(p1)(p2)(p5)(p6)(3\;4\;7\;8)+\text{CP}]\\ -32(p^2-m^2)[(p1)(p2)(p3)(p4)(p5)(p6)(7\;8)+\text{CP}]+128(p1)(p2)\\ \ldots(p8).\end{aligned}$$

It was not possible to evaluate this trace on a Telefunken TR 440 computer with 80 K using REDUCE in a straightforward way. Moreover, symbolic computation programs cannot easily extract factors as $(p^2-m^2)^n$ from an expression. Thus the output, if available, would consist of thousands of terms and not be very useful.

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